

# Post-Newtonian Computation of Binary Inspirational Waveforms\*

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## I. INTRODUCTION

Astrophysical systems known as inspiralling compact binaries are among the most interesting sources to hunt for gravitational radiation in the future network of laser-interferometric detectors, composed of the large-scale interferometers VIRGO and LIGO, and the medium-scale ones GEO and TAMA (see the books [1–3] for reviews, and the contribution of B Schutz in this volume). These systems are composed of two compact objects, i.e. gravitationally-condensed neutron stars or black holes, whose orbit follows an inward spiral, with decreasing orbital radius  $r$  and increasing orbital frequency  $\omega$ . The inspiral is driven by the loss of energy associated with the gravitational-wave emission. Because the dynamics of a binary is essentially aspherical, inspiralling compact binaries are strong emitters of gravitational radiation. Tidal interactions between the compact objects are expected to play a little role during most of the inspiral phase; the mass transfer (in the case of neutron stars) does not occur until very late, near the final coalescence. Inspiralling compact binaries are very clean systems, essentially dominated by gravitational forces. Therefore, the relevant model for describing the inspiral phase consists of two point-masses moving under their mutual gravitational attraction. As a simplification for the theoretical analysis, the orbit of inspiralling binaries can be considered to be circular, apart from the gradual inspiral, with a good approximation. At some point in the evolution, there will be a transition from the adiabatic inspiral to the plunge of the two objects followed by the collision and final merger. Evidently the model of point-masses breaks down at this point, and is to be replaced by a fully relativistic numerical computation of the plunge and merger (see the contribution of E Seidel in this volume).

Currently the theoretical prediction from general relativity for the gravitational waves emitted during the inspiral phase is determined using the post-Newtonian approximation (see [4,5] for reviews). This is possible because the dynamics of inspiralling compact binaries, though very relativistic, is not *fully* relativistic: the orbital velocity  $v$  is always less than one third of  $c$  (say). However, because  $1/3$  is far from negligible as compared to 1, the gravitational-radiation waveform should be predicted up to a high post-Newtonian order. In particular, the radiation *reaction* onto the orbit, which triggers the inspiral, is to be determined with the maximal precision, corresponding to at least the second and maybe the third post-Newtonian (3PN, or  $1/c^6$ ) order [6,7]. Notice that the zeroth order in this post-Newtonian counting corresponds to the dominant radiation reaction force (already of the order of 2.5PN relative to the Newtonian force), which is due to the change in the quadrupole moment of the source. Actually, the method is not to compute directly the radiation reaction force but to determine the inspiral rate from the energy balance equation relating the mechanical loss of energy in the binary's centre of mass to the total emitted flux at infinity.

The implemented strategy is to develop a formalism for the emission and propagation of gravitational waves from a general isolated system, and only then, once some general formulae valid to some prescribed post-Newtonian order are in our hands, to apply the formalism to compact binaries. Hence, we consider in this paper a particular formalism applicable to a general description of matter, under the tenet of validity of the post-Newtonian expansion, namely that the matter should be slowly moving, weakly stressed and self-gravitating. Within this formalism we compute the retarded far field of the source by means of a formal post-Minkowskian expansion, valid in the exterior of the source, and parametrized by some appropriately defined multipole moments describing the source. From the post-Minkowskian expansion we obtain a relation (correct up to the prescribed post-Newtonian order) between the *radiative* multipole moments parametrizing the metric field at infinity, and the source multipole moments. On the other hand, the source multipole moments are obtained as some specific integrals extending over the distribution of matter fields in the source and the contribution of the gravitational field itself. The source moments are computed separately up to the same post-Newtonian order. The latter formalism has been developed by Blanchet, Damour and Iyer [8–14]. More recently, a different formalism has been proposed and implemented by Will and Wiseman [15] (see also [16,17]). The two formalisms are equivalent at the most general level, but the details of the computations are quite far apart. In the

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second stage, one applies the formalism to a system of point-particles (modelling compact objects) by substituting for the matter stress-energy tensor that expression, involving delta-functions, which is appropriate for point-particles. This entails some divergencies due to the infinite self-field of point-particles. Our present method is to cure them systematically by means of a variant of the Hadamard regularization (based on the concept of ‘partie finie’) [18,19].

In this paper, we first analyse the binary inspiral gravitational waveform at the simplest Newtonian approximation. Notably, we spend some time describing the relative orientation of the binary with respect to the detector. Then we compute, still at the ‘Newtonian order’ (corresponding, in fact, to the quadrupole approximation), the evolution in the course of time of the orbital phase of the binary, which is a crucial quantity to predict. Next, we review the main steps of our general wave-generation formalism, with emphasis on the definition of the various types of multipole moments which are involved. At last, we present the result for the binary inspiral waveform whose current post-Newtonian precision is 2PN in the wave amplitude and 2.5PN in the orbital phase (that is  $1/c^5$  beyond the quadrupole radiation reaction). However, since our ultimate aim is to construct accurate templates to be used in the data analysis of detectors, it is appropriate to warm up with a short review of the optimal filtering technique which will be used for hunting the inspiral binary waveform (see [20] for an extended review).

## II. SUMMARY OF OPTIMAL SIGNAL FILTERING

Let  $o(t)$  be the raw output of the detector, which is made of the superposition of the useful gravitational-wave signal  $h(t)$  and of noise  $n(t)$ :

$$o(t) = h(t) + n(t). \quad (2.1)$$

The noise is assumed to be a stationary Gaussian random variable, with zero expectation value,

$$\overline{n(t)} = 0, \quad (2.2)$$

and with (supposedly known) frequency-dependent power spectral density  $S_n(\omega)$  satisfying

$$\overline{\tilde{n}(\omega)\tilde{n}^*(\omega')} = 2\pi\delta(\omega - \omega')S_n(\omega), \quad (2.3)$$

where  $\tilde{n}(\omega)$  is the Fourier transform of  $n(t)$ . In (2.2) and (2.3), we denote by an upper bar the average over many realizations of noise in a large ensemble of detectors. From (2.3), we have  $S_n(\omega) = S_n^*(\omega) = S_n(-\omega) > 0$ .

Looking for the signal  $h(t)$  in the output of the detector  $o(t)$ , the experimenters construct the correlation  $c(t)$  between  $o(t)$  and a filter  $q(t)$ , i.e.

$$c(t) = \int_{-\infty}^{+\infty} dt' o(t')q(t+t'), \quad (2.4)$$

and divide  $c(t)$  by the square root of its variance (or correlation noise). Thus, the experimenters consider the ratio

$$\sigma[q](t) = \frac{c(t)}{(\overline{c^2(t)} - \overline{c(t)}^2)^{1/2}} = \frac{\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \tilde{o}(\omega)\tilde{q}^*(\omega)e^{i\omega t}}{\left(\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_n(\omega)|\tilde{q}(\omega)|^2\right)^{1/2}}, \quad (2.5)$$

where  $\tilde{o}(\omega)$  and  $\tilde{q}(\omega)$  are the Fourier transforms of  $o(t)$  and  $q(t)$ . The expectation value (or ensemble average) of this ratio defines the filtered signal-to-noise ratio

$$\rho[q](t) = \overline{\sigma[q](t)} = \frac{\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \tilde{h}(\omega)\tilde{q}^*(\omega)e^{i\omega t}}{\left(\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_n(\omega)|\tilde{q}(\omega)|^2\right)^{1/2}}. \quad (2.6)$$

The optimal filter (or Wiener filter) which maximizes the signal-to-noise (2.6) at a particular instant  $t = 0$  (say), is given by the matched filtering theorem as

$$\tilde{q}(\omega) = \gamma \frac{\tilde{h}(\omega)}{S_n(\omega)}, \quad (2.7)$$

where  $\gamma$  is an arbitrary real constant. The optimal filter (2.7) is matched on the expected signal  $\tilde{h}(\omega)$  itself, and weighted by the inverse of the power spectral density of the noise. The maximum signal to noise, corresponding to the optimal filter (2.7), is given by

$$\rho = \left( \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{|\tilde{h}(\omega)|^2}{S_n(\omega)} \right)^{1/2} = \langle h, h \rangle^{1/2}. \quad (2.8)$$

This is the best achievable signal-to-noise ratio with a linear filter. In (2.8), we have used, for any two real functions  $f(t)$  and  $g(t)$ , the notation

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\tilde{f}(\omega) \tilde{g}^*(\omega)}{S_n(\omega)} \quad (2.9)$$

for an inner scalar product satisfying  $\langle f, g \rangle = \langle f, g \rangle^* = \langle g, f \rangle$ .

In practice, the signal  $h(t)$  or  $\tilde{h}(\omega)$  is of known form (given, for instance, by (3.9)–(3.14) later) but depends on an unknown set of parameters which describe the source of radiation, and are to be measured. The experimenters must therefore use a whole family of filters analogous to (2.7) but in which the signal is parametrized by a whole family of ‘test’ parameters which are *a priori* different from the actual source parameters. Thus, one will have to define and use a lattice of filters in the parameter space. The set of parameters maximizing the signal to noise (2.6) is equal, by the matched filtering theorem, to the set of source parameters. However, in a single detector, the experimenters maximize the ratio (2.5) rather than the signal to noise (2.6), and therefore make errors on the determination of the parameters, depending on a particular realization of noise in the detector. If the signal-to-noise ratio is high enough, the measured values of the parameters are Gaussian distributed around the source parameters, with variances and correlation coefficients given by the covariance matrix, the computation of which we now recall. Since the optimal filter (2.7) is defined up to an arbitrary multiplicative constant, it is convenient to treat separately a constant amplitude parameter in front of the signal (involving, in general, the distance of the source). We shall thus write the signal in the form

$$\tilde{h}(\omega; A, \lambda_a) = A \tilde{k}(\omega; \lambda_a), \quad (2.10)$$

where  $A$  denotes some amplitude parameter. The function  $\tilde{k}$  depends only on the other parameters, collectively denoted by  $\lambda_a$  where the label  $a$  ranges on the values  $1, \dots, N$ . The family of matched filters (or ‘templates’) we consider is defined by

$$\tilde{q}(\omega; {}_t\lambda_a) = \gamma' \frac{\tilde{k}(\omega; {}_t\lambda_a)}{S_n(\omega)}, \quad (2.11)$$

where  ${}_t\lambda_a$  is a set of test parameters, assumed to be all independent, and  $\gamma'$  is arbitrary. By substituting (2.11) into (2.5) and choosing  $t = 0$ , we get, with the notation of (2.9),

$$\sigma({}_t\lambda) = \frac{\langle o, k({}_t\lambda) \rangle}{\langle k({}_t\lambda), k({}_t\lambda) \rangle^{1/2}}. \quad (2.12)$$

(Note that  $\sigma$  is in fact a function of both the parameters  $\lambda_a$  and  ${}_t\lambda_a$ .) Now the experimenters choose as their best estimate of the source parameters  $\lambda_a$  the *measured* parameters  ${}_m\lambda_a$  which among all the test parameters  ${}_t\lambda_a$  (independently) maximize (2.12), i.e. which satisfy

$$\frac{\partial \sigma}{\partial {}_t\lambda_a}({}_m\lambda) = 0, \quad a = 1, \dots, N. \quad (2.13)$$

Assuming that the signal to noise is high enough, we can work out (2.13) up to the first order in the difference between the actual source parameters and the measured ones,

$$\delta\lambda_a = \lambda_a - {}_m\lambda_a. \quad (2.14)$$

As a result, we obtain

$$\delta\lambda_a = C_{ab} \left\{ -\langle n, \frac{\partial h}{\partial \lambda_b} \rangle + \frac{\langle n, h \rangle}{\langle h, h \rangle} \langle h, \frac{\partial h}{\partial \lambda_b} \rangle \right\}, \quad (2.15)$$

where a summation is understood on the dummy label  $b$ , and where the matrix  $\mathcal{C}_{ab}$  (with  $a, b = 1, \dots, N$ ) is the inverse of the Fisher information matrix

$$\mathcal{D}_{ab} = \left\langle \frac{\partial h}{\partial \lambda_a}, \frac{\partial h}{\partial \lambda_b} \right\rangle - \frac{1}{\langle h, h \rangle} \left\langle h, \frac{\partial h}{\partial \lambda_a} \right\rangle \left\langle h, \frac{\partial h}{\partial \lambda_b} \right\rangle \quad (2.16)$$

(we have  $\mathcal{C}_{ab}\mathcal{D}_{bc} = \delta_{ac}$ ). On the right-hand sides of (2.15) and (2.16), the signal is equally (with this approximation) parametrized by the measured or actual parameters. Since the noise is Gaussian, so are, by (2.15), the variables  $\delta\lambda_a$  (indeed,  $\delta\lambda_a$  result from a linear operation on the noise variable). The expectation value and quadratic moments of the distribution of these variables are readily obtained from the facts that  $\overline{\langle n, f \rangle} = 0$  and  $\overline{\langle n, f \rangle \langle n, g \rangle} = \langle f, g \rangle$  for any deterministic functions  $f$  and  $g$  (see (2.2) and (2.3)). We then obtain

$$\overline{\delta\lambda_a} = 0, \quad \overline{\delta\lambda_a \delta\lambda_b} = \mathcal{C}_{ab}. \quad (2.17)$$

Thus, the matrix  $\mathcal{C}_{ab}$  (the inverse of (2.16)) is the matrix of variances and correlation coefficients, or covariance matrix, of the variables  $\delta\lambda_a$ . The probability distribution of  $\delta\lambda_a$  reads as

$$P(\delta\lambda_a) = \frac{1}{\sqrt{(2\pi)^{N+1} \det \mathcal{C}}} \exp \left\{ -\frac{1}{2} \mathcal{D}_{ab} \delta\lambda_a \delta\lambda_b \right\}, \quad (2.18)$$

where  $\det \mathcal{C}$  is the determinant of  $\mathcal{C}_{ab}$ . A similar analysis can be done for the measurement of the amplitude parameter  $A$  of the signal.

### III. NEWTONIAN BINARY POLARIZATION WAVEFORMS

The source of gravitational waves is a binary system made of two point-masses moving on a circular orbit. We assume that the masses do not possess any intrinsic spins, so that the motion of the binary takes place in a plane. To simplify the presentation we suppose that the centre of mass of the binary is at rest with respect to the detector. The detector is a large-scale laser-interferometric detector like VIRGO or LIGO, with two perpendicular arms (with length 3 km in the case of VIRGO). The two laser beams inside the arms are separated by the beam-splitter which defines the central point of the interferometer. We introduce an orthonormal right-handed triad  $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$  linked with the detector, with  $\mathbf{X}$  and  $\mathbf{Y}$  pointing along the two arms of the interferometer, and  $\mathbf{Z}$  pointing toward the zenithal direction. We denote by  $\mathbf{n}$  the direction of the detector as seen from the source, that is,  $-\mathbf{n}$  is defined as the unit vector pointing from the centre of the interferometer to the binary's centre of mass. We introduce some spherical angles  $\alpha$  and  $\beta$  such that

$$-\mathbf{n} = \mathbf{X} \sin \alpha \cos \beta + \mathbf{Y} \sin \alpha \sin \beta + \mathbf{Z} \cos \alpha. \quad (3.1)$$

Thus, the plane  $\beta = \text{constant}$  defines the plane which is vertical, as seen from the detector, and which contains the source. Next, we introduce an orthonormal right-handed triad  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  which is linked to the binary's orbit, with  $\mathbf{x}$  and  $\mathbf{y}$  located in the orbital plane, and  $\mathbf{z}$  along the normal to the orbital plane. The vector  $\mathbf{x}$  is chosen to be perpendicular to  $\mathbf{n}$ ; thus,  $\mathbf{n}$  is within the plane formed by  $\mathbf{y}$  and  $\mathbf{z}$ . The orientation of this triad is 'right-hand' with respect to the sense of motion. We denote by  $i$  the inclination angle, namely the angle between the direction of the source or line-of-sight  $\mathbf{n}$  and the normal  $\mathbf{z}$  to the orbital plane. Since  $\mathbf{z}$  is right-handed with respect to the sense of motion we have  $0 \leq i \leq \pi$ . Furthermore, we define two unit vectors  $\mathbf{p}$  and  $\mathbf{q}$ , called the polarization vectors, in the plane orthogonal to  $\mathbf{n}$  (or plane of the sky). We choose  $\mathbf{p} = \mathbf{x}$  and define  $\mathbf{q}$  in such a way that the triad  $(\mathbf{n}, \mathbf{p}, \mathbf{q})$  is right-handed; thus

$$\mathbf{n} = \mathbf{y} \sin i + \mathbf{z} \cos i, \quad (3.2)$$

$$\mathbf{p} = \mathbf{x}, \quad (3.3)$$

$$\mathbf{q} = \mathbf{y} \cos i - \mathbf{z} \sin i. \quad (3.4)$$

Notice that the direction  $\mathbf{p} \equiv \mathbf{x}$  is one of the 'ascending node'  $N$  of the binary, namely the point at which the bodies cross the plane of the sky moving toward the detector. Thus, the polarization vectors  $\mathbf{p}$  and  $\mathbf{q}$  lie, respectively, along the major and minor axis of the projection onto the plane of the sky of the (circular) orbit, with  $\mathbf{p}$  pointing toward  $N$  using the standard practice of celestial mechanics. Finally, let us denote by  $\xi$  the polarization angle between  $\mathbf{p}$  and the vertical plane  $\beta = \text{constant}$ ; that is,  $\xi$  is the angle between the vertical and the direction of the node  $N$ . We have

$$\mathbf{n} = -\mathbf{X} \sin \alpha \cos \beta - \mathbf{Y} \sin \alpha \sin \beta - \mathbf{Z} \cos \alpha, \quad (3.5)$$

$$\begin{aligned} \mathbf{p} = & \mathbf{X}(\cos \xi \cos \alpha \cos \beta + \sin \xi \sin \beta) \\ & + \mathbf{Y}(\cos \xi \cos \alpha \sin \beta - \sin \xi \cos \beta) - \mathbf{Z} \cos \xi \sin \alpha, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \mathbf{q} = & \mathbf{X}(-\sin \xi \cos \alpha \cos \beta + \cos \xi \sin \beta) \\ & + \mathbf{Y}(-\sin \xi \cos \alpha \sin \beta - \cos \xi \cos \beta) + \mathbf{Z} \sin \xi \sin \alpha. \end{aligned} \quad (3.7)$$

Defining all these angles, the relative orientation of the binary with respect to the interferometric detector is entirely determined. Indeed using (3.4) and (3.7) one relates the triad  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  associated with the source to the triad  $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$  linked with the detector.

The gravitational wave as it propagates through the detector in the wave zone of the source is described by the so-called transverse and traceless (TT) asymptotic waveform  $h_{ij}^{\text{TT}} = (g_{ij} - \delta_{ij})^{\text{TT}}$ , where  $g_{ij}$  denotes the spatial covariant metric in a coordinate system adapted to the wave zone, and  $\delta_{ij}$  is the Kronecker metric. Neglecting terms dying out like  $1/R^2$  in the distance to the source, the two polarization states of the wave, customarily denoted  $h_+$  and  $h_\times$ , are given by

$$h_+ = \frac{1}{2}(p_i p_j - q_i q_j) h_{ij}^{\text{TT}}, \quad (3.8)$$

$$h_\times = \frac{1}{2}(p_i q_j + p_j q_i) h_{ij}^{\text{TT}}, \quad (3.9)$$

where  $p_i$  and  $q_i$  are the components of the polarization vectors. The detector is directly sensitive to a linear combination of the polarization waveforms  $h_+$  and  $h_\times$  given by

$$h(t) = F_+ h_+(t) + F_\times h_\times(t), \quad (3.10)$$

where  $F_+$  and  $F_\times$  are the so-called beam-pattern functions of the detector, which are some given functions (for a given type of detector) of the direction of the source  $\alpha, \beta$  and of the polarization angle  $\xi$ . This  $h(t)$  is the gravitational-wave signal looked for in the data analysis of section II, and used to construct the optimal filter (2.10). In the case of the laser-interferometric detector we have

$$F_+ = \frac{1}{2}(1 + \cos^2 \alpha) \cos 2\beta \cos 2\xi + \cos \alpha \sin 2\beta \sin 2\xi, \quad (3.11)$$

$$F_\times = -\frac{1}{2}(1 + \cos^2 \alpha) \cos 2\beta \sin 2\xi + \cos \alpha \sin 2\beta \cos 2\xi. \quad (3.12)$$

The orbital plane and the direction of the node  $N$  are fixed so the polarization angle  $\xi$  is constant (in the case of spinning particles, the orbital plane precesses around the direction of the total angular momentum, and angle  $\xi$  varies). Thus, the gravitational wave  $h(t)$  depends on time only through the two polarization waveforms  $h_+(t)$  and  $h_\times(t)$ . In turn, these waveforms depend on time through the binary's orbital phase  $\phi(t)$  and the orbital frequency  $\omega(t) = d\phi(t)/dt$ . The orbital phase is defined as the angle, oriented in the sense of motion, between the ascending node  $N$  and the direction of one of the particles, conventionally particle 1 (thus  $\phi = 0$  modulo  $2\pi$  when the two particles lie along  $\mathbf{p}$ , with particle 1 at the ascending node). In the absence of any radiation reaction, the orbital frequency would be constant, and so the phase would evolve linearly with time. Because of the radiation reaction forces, the actual variation of  $\phi(t)$  is nonlinear, and the orbit spirals in and shrinks to zero-size to account, via the Kepler third law, for the gravitational-radiation energy loss. The main problem of the construction of accurate templates for the detection of inspiralling compact binaries is the prediction of the time variation of the phase  $\phi(t)$ . Indeed, because of the accumulation of cycles, most of the accessible information allowing accurate measurements of the binary's intrinsic parameters (such as the two masses) is contained within the phase, and rather less accurate information is available in the wave amplitude itself. For instance, the relative precision in the determination of the distance  $R$  to the source, which affects the wave amplitude, is less than for the masses, which strongly affect the phase evolution [6,7]. Hence, we can often neglect the higher-order contributions to the amplitude, which means retaining only the dominant harmonics in the waveform, which corresponds to a frequency at twice the orbital frequency.

Once the functions  $\phi(t)$  and  $\omega(t)$  are known they must be inserted into the polarization waveforms computed by means of some wave-generation formalism. For instance, using the quadrupole formalism, which neglects all the harmonics but the dominant one, we find

$$h_+ = -\frac{2G\mu}{c^2 R} \left( \frac{Gm\omega}{c^3} \right)^{2/3} (1 + \cos^2 i) \cos 2\phi, \quad (3.13)$$

$$h_\times = -\frac{2G\mu}{c^2 R} \left( \frac{Gm\omega}{c^3} \right)^{2/3} (2 \cos i) \sin 2\phi \quad (3.14)$$

where  $R$  denotes the absolute luminosity distance of the binary's centre of mass; the mass parameters are given by

$$m = m_1 + m_2; \quad \mu = \frac{m_1 m_2}{m}; \quad \nu = \frac{\mu}{m}. \quad (3.15)$$

This last parameter  $\nu$ , introduced for later convenience, is the ratio between the reduced mass and the total mass, and is such that  $0 < \nu \leq 1/4$  with  $\nu \rightarrow 0$  in the test-mass limit and  $\nu = 1/4$  in the case of two equal masses.

#### IV. NEWTONIAN ORBITAL PHASE EVOLUTION

Let  $\mathbf{y}_1(t)$  and  $\mathbf{y}_2(t)$  be the two trajectories of the masses  $m_1$  and  $m_2$ , and  $\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$  be their relative position, and denote  $r = |\mathbf{y}|$ . The velocities are  $\mathbf{v}_1(t) = d\mathbf{y}_1/dt$ ,  $\mathbf{v}_2(t) = d\mathbf{y}_2/dt$  and  $\mathbf{v}(t) = d\mathbf{y}/dt$ . The Newtonian equations of motion read as

$$\frac{d\mathbf{v}_1}{dt} = -\frac{Gm_2}{r^3}\mathbf{y}; \quad \frac{d\mathbf{v}_2}{dt} = \frac{Gm_1}{r^3}\mathbf{y}. \quad (4.1)$$

The difference between these two equations yields the relative acceleration,

$$\frac{d\mathbf{v}}{dt} = -\frac{Gm}{r^3}\mathbf{y}. \quad (4.2)$$

We place ourselves into the Newtonian centre-of-mass frame defined by

$$m_1\mathbf{y}_1 + m_2\mathbf{y}_2 = \mathbf{0}, \quad (4.3)$$

in which frame the individual trajectories  $\mathbf{y}_1$  and  $\mathbf{y}_2$  are related to the relative one  $\mathbf{y}$  by

$$\mathbf{y}_1 = \frac{m_2}{m}\mathbf{y}; \quad \mathbf{y}_2 = -\frac{m_1}{m}\mathbf{y}. \quad (4.4)$$

The velocities are given similarly by

$$\mathbf{v}_1 = \frac{m_2}{m}\mathbf{v}; \quad \mathbf{v}_2 = -\frac{m_1}{m}\mathbf{v}. \quad (4.5)$$

In principle, the binary's phase evolution  $\phi(t)$  should be determined from a knowledge of the radiation reaction forces acting locally on the orbit. At the Newtonian order, this means considering the 'Newtonian' radiation reaction force, which is known to contribute to the total acceleration only at the 2.5PN level, i.e.  $1/c^5$  smaller than the Newtonian acceleration (where  $5 = 2s + 1$ , with  $s = 2$  the helicity of the graviton). A simpler computation of the phase is to deduce it from the energy balance equation between the loss of centre-of-mass energy and the total flux emitted at infinity in the form of waves. In the case of circular orbits one needs only to find the decrease of the orbital separation  $r$  and for that purpose the balance of energy is sufficient. Relying on an energy balance equation is the method we follow for computing the phase of inspiralling binaries in higher post-Newtonian approximations (see section VI). Thus, we write

$$\frac{dE}{dt} = -\mathcal{L}, \quad (4.6)$$

where  $E$  is the centre-of-mass energy, given at the Newtonian order by

$$E = -\frac{Gm_1 m_2}{2r}, \quad (4.7)$$

and where  $\mathcal{L}$  denotes the total energy flux (or gravitational 'luminosity'), deduced to the Newtonian order from the quadrupole formula of Einstein:

$$\mathcal{L} = \frac{G}{5c^5} \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3}. \quad (4.8)$$

The quadrupole moment is merely the Newtonian (trace-free) quadrupole of the source, which reads in the case of the point-particle binary as

$$Q_{ij} = m_1(y_1^i y_1^j - \frac{1}{3}\delta^{ij} \mathbf{y}_1^2) + 1 \leftrightarrow 2. \quad (4.9)$$

In the mass-centred frame (4.3) we get

$$Q_{ij} = \mu(y^i y^j - \frac{1}{3}\delta^{ij} r^2). \quad (4.10)$$

The third time derivative of  $Q_{ij}$  needed in the quadrupole formula (4.8) is easily obtained. When an acceleration is generated we replace it by the Newtonian equation of motion (4.2). In the case of a circular orbit we get

$$\frac{d^3 Q_{ij}}{dt^3} = -4 \frac{Gm\mu}{r^3} (y^i v^j + y^j v^i) \quad (4.11)$$

(this is automatically trace-free because  $\mathbf{y} \cdot \mathbf{v} = 0$ ). Replacing (4.11) into (4.8) leads to the ‘Newtonian’ flux

$$\mathcal{L} = \frac{32}{5} \frac{G^3 m^2 \mu^2}{c^5 r^4} \mathbf{v}^2. \quad (4.12)$$

A better way to express the flux is in terms of some dimensionless quantities, namely the mass ratio  $\nu$  given in (3.15), and a very convenient post-Newtonian parameter defined from the orbital frequency  $\omega$  by

$$x = \left( \frac{Gm\omega}{c^3} \right)^{2/3}. \quad (4.13)$$

Notice that  $x$  is of formal order  $O(1/c^2)$  in the post-Newtonian expansion. Thanks to the Kepler law  $Gm = r^3 \omega^2$  we transform (4.12) and arrive at

$$\mathcal{L} = \frac{32}{5} \frac{c^5}{G} \nu^2 x^5. \quad (4.14)$$

In this form the only factor having a dimension is

$$\frac{c^5}{G} \approx 3.63 \times 10^{52} \text{ W}, \quad (4.15)$$

which is the Planck unit of a power, which turns out to be independent of the Planck constant. (Notice that instead of  $c^5/G$  the inverse ratio  $G/c^5$  appears as a factor in the quadrupole formula (4.8).) On the other hand, we find that  $E$  reads simply

$$E = -\frac{1}{2} \mu c^2 x. \quad (4.16)$$

Next we replace (4.14) and (4.16) into the balance equation (4.6), and find in this way an ordinary differential equation which is easily integrated for the unknown  $x$ . We introduce for later convenience the dimensionless time variable

$$\tau = \frac{c^3 \nu}{5Gm} (t_c - t), \quad (4.17)$$

where  $t_c$  is a constant of integration. Then the solution reads

$$x(t) = \frac{1}{4} \tau^{-1/4}. \quad (4.18)$$

It is clear that  $t_c$  represents the instant of coalescence, at which (by definition) the orbital frequency diverges to infinity. Then a further integration yields  $\phi = \int \omega dt = -\frac{5}{\nu} \int x^{2/3} d\tau$ , and we get the looked for result

$$\phi_c - \phi(t) = \frac{1}{\nu} \tau^{5/8}, \quad (4.19)$$

where  $\phi_c$  denotes the constant phase at the instant of coalescence. It is often useful to consider the number  $\mathcal{N}$  of gravitational-wave cycles which are left until the final coalescence starting from some frequency  $\omega$ :

$$\mathcal{N} = \frac{\phi_c - \phi}{\pi} = \frac{1}{32\pi\nu} x^{-5/2}. \quad (4.20)$$

As we see the post-Newtonian order of magnitude of  $\mathcal{N}$  is  $c^{+5}$ , that is the inverse of the order  $c^{-5}$  of radiation reaction effects. As a matter of fact,  $\mathcal{N}$  is a large number, approximately equal to  $1.6 \times 10^4$  in the case of two neutron stars between 10 and 1000 Hz (roughly the frequency bandwidth of the detector VIRGO). Data analysts of detectors have estimated that, in order not to suffer a too severe reduction of signal to noise, one should monitor the phase evolution with an accuracy comparable to one gravitational-wave cycle (i.e.  $\delta\mathcal{N} \sim 1$ ) or better. Now it is clear, from a post-Newtonian point of view, that since the ‘Newtonian’ number of cycles given by (4.20) is formally of order  $c^{+5}$ , any post-Newtonian correction therein which is larger than order  $c^{-5}$  is expected to contribute to the phase evolution more than that allowed by the previous estimate. Therefore, one expects that in order to construct accurate templates it will be necessary to include into the phase the post-Newtonian corrections up to at least the 2.5PN or  $1/c^5$  order. This expectation has been confirmed by various studies [21–24] which showed that in advanced detectors the 2.5PN or, better, the 3PN approximation is required in the case of inspiralling neutron star binaries. Notice that 3PN here means 3PN in the centre-of-mass energy  $E$ , which is deduced from the 3PN equations of motion, as well as in the total flux  $\mathcal{L}$ , which is computed from a 3PN wave-generation formalism. For the moment the phase has been completed to the 2.5PN order [25–27,15]; the 3PN order is still incomplete (but, see [13,28,29]).

## V. POST-NEWTONIAN WAVE GENERATION

### A. Field equations

We consider a general compact-support stress-energy tensor  $T^{\mu\nu}$  describing the isolated source, and we look for the solutions, in the form of a (formal) post-Newtonian expansion, of the Einstein field equations,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}T^{\mu\nu}, \quad (5.1)$$

and thus also of their consequence, the equations of motion  $\nabla_\nu T^{\mu\nu} = 0$  of the source. We impose the condition of harmonic coordinates, i.e. the gauge condition

$$\partial_\nu h^{\mu\nu} = 0; \quad h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \quad (5.2)$$

where  $g$  and  $g^{\mu\nu}$  denote the determinant and inverse of the covariant metric  $g_{\mu\nu}$ , and where  $\eta^{\mu\nu}$  is a Minkowski metric:  $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ . Then the Einstein field equations (5.1) can be replaced by the so-called *relaxed* equations, which take the form of simple wave equations,

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}, \quad (5.3)$$

where the box operator is the flat d’Alembertian  $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$ , and where the source term  $\tau^{\mu\nu}$  can be viewed as the stress-energy pseudotensor of the matter and gravitational fields in harmonic coordinates. It is given by

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}. \quad (5.4)$$

$\tau^{\mu\nu}$  is not a generally-covariant tensor, but only a Lorentz tensor relative to the Minkowski metric  $\eta_{\mu\nu}$ . As a consequence of the gauge condition (5.2),  $\tau^{\mu\nu}$  is conserved in the usual sense,

$$\partial_\nu \tau^{\mu\nu} = 0 \quad (5.5)$$

(this is equivalent to  $\nabla_\nu T^{\mu\nu} = 0$ ). The gravitational source term  $\Lambda^{\mu\nu}$  is a quite complicated, highly nonlinear (quadratic at least) functional of  $h^{\mu\nu}$  and its first- and second-spacetime derivatives.

We supplement the resolution of the field equations (5.2) and (5.3) by the requirement that the source does not receive any radiation from other sources located very far away. Such a requirement of ‘no-incoming radiation’ is to be imposed at Minkowskian past null infinity (taking advantage of the presence of the Minkowski metric  $\eta_{\mu\nu}$ ); this corresponds to the limit  $r = |\mathbf{x}| \rightarrow +\infty$  with  $t + r/c = \text{constant}$ . (Please do not confuse this  $r$  with the same  $r$  denoting the separation between the two bodies in section IV.) The precise formulation of the no-incoming radiation condition is

$$\lim_{\substack{r \rightarrow +\infty \\ t + \frac{r}{c} = \text{constant}}} \left[ \frac{\partial}{\partial r}(rh^{\mu\nu}) + \frac{\partial}{c\partial t}(rh^{\mu\nu}) \right] (\mathbf{x}, t) = 0. \quad (5.6)$$



In addition,  $r\partial_\lambda h^{\mu\nu}$  should be bounded in the same limit. Actually we often adopt, for technical reasons, the more restrictive condition that the field is stationary before some finite instant  $-\mathcal{T}$  in the past (refer to [8] for details). With the no-incoming radiation condition (5.5) or (5.6) we transform the differential Einstein equation (5.3) into the equivalent integro-differential system

$$h^{\mu\nu} = \frac{16\pi G}{c^4} \square_R^{-1} \tau^{\mu\nu}, \quad (5.7)$$

where  $\square_R^{-1}$  denotes the standard retarded inverse d'Alembertian given by

$$(\square_R^{-1} \tau)(\mathbf{x}, t) = -\frac{1}{4\pi} \int \frac{d^3 \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \tau(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c). \quad (5.8)$$

## B. Source moments

In this section we shall solve the field equations (5.2) and (5.3) in the exterior of the isolated source by means of a multipole expansion, parametrized by some appropriate source multipole moments. The particularity of the moments we shall obtain, is that they are defined from the *formal* post-Newtonian expansion of the pseudotensor  $\tau^{\mu\nu}$ , supposing that the latter expansion can be iterated to any order. Therefore, these source multipole moments are physically valid only in the case of a slowly-moving source (slow internal velocities; weak stresses). The general structure of the post-Newtonian expansion involves besides the usual powers of  $1/c$  some arbitrary powers of the logarithm of  $c$ , say

$$\bar{\tau}^{\mu\nu}(t, \mathbf{x}, c) = \sum_{p,q} \frac{(\ln c)^q}{c^p} \tau_{pq}^{\mu\nu}(t, \mathbf{x}), \quad (5.9)$$

where the overbar denotes the formal post-Newtonian expansion, and where  $\tau_{pq}^{\mu\nu}$  are the functional coefficients of the expansion ( $p, q$  are integers, including zero). Now, the general multipole expansion of the metric field  $h^{\mu\nu}$ , denoted by  $\mathcal{M}(h^{\mu\nu})$ , is found by requiring that when re-developed into the near-zone, i.e. in the limit where  $r/c \rightarrow 0$  (this is equivalent with the formal re-expansion when  $c \rightarrow \infty$ ), it *matches* with the multipole expansion of the post-Newtonian expansion  $\bar{h}^{\mu\nu}$  (whose structure is similar to (5.9)) in the sense of the mathematical technics of matched asymptotic expansions. We find [11,14] that the multipole expansion  $\mathcal{M}(h^{\mu\nu})$  satisfying the matching is uniquely determined, and is composed of the sum of two terms,

$$\mathcal{M}(h^{\mu\nu}) = \text{finite part } \square_R^{-1}[\mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left\{ \frac{1}{r} \mathcal{F}_L^{\mu\nu}(t - r/c) \right\}. \quad (5.10)$$

The first term, in which  $\square_R^{-1}$  is the flat retarded operator (5.8), is a particular solution of the Einstein field equations in vacuum (outside the source), i.e. it satisfies  $\square h_{\text{part}}^{\mu\nu} = \mathcal{M}(\Lambda^{\mu\nu})$ . The second term is a retarded solution of the source-free homogeneous wave equation, i.e.  $\square h_{\text{hom}}^{\mu\nu} = 0$ . We denote  $\partial_L = \partial_{i_1} \dots \partial_{i_l}$  where  $L = i_1 \dots i_l$  is a multi-index composed of  $l$  indices; the  $l$  summations over the indices  $i_1 \dots i_l$  are not indicated in (5.10). The ‘multipole moments’ parametrizing this homogeneous solution are given explicitly by (with  $u = t - r/c$ )

$$\mathcal{F}_L^{\mu\nu}(u) = \text{finite part} \int d^3 \mathbf{x} \hat{x}_L \int_{-1}^1 dz \delta_l(z) \bar{\tau}^{\mu\nu}(\mathbf{x}, u + z|\mathbf{x}|/c), \quad (5.11)$$

where the integrand contains the *post-Newtonian* expansion of the pseudostress-energy tensor  $\bar{\tau}^{\mu\nu}$ , whose structure reads like (5.9). In (5.11), we denote the symmetric-trace-free (STF) projection of the product of  $l$  vectors  $x^i$  with a hat, so that  $\hat{x}_L = \text{STF}(x^L)$ , with  $x^L = x^{i_1} \dots x^{i_l}$  and  $L = i_1 \dots i_l$ ; for instance,  $\hat{x}_{ij} = x_i x_j - \frac{1}{3} \delta_{ij} \mathbf{x}^2$ . The function  $\delta_l(z)$  is given by

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}l!} (1-z^2)^l, \quad (5.12)$$

and satisfies the properties

$$\int_{-1}^1 dz \delta_l(z) = 1; \quad \lim_{l \rightarrow +\infty} \delta_l(z) = \delta(z) \quad (5.13)$$

(where  $\delta(z)$  is the Dirac measure). Both terms in (5.10) involve an operation of taking a finite part. This finite part can be defined precisely by means of an analytic continuation (see [14] for details), but it is in fact basically equivalent to taking the finite part of a divergent integral in the sense of Hadamard [18]. Notice, in particular, that the finite part in the expression of the multipole moments (5.11) deals with the behaviour of the integral *at infinity*:  $r \rightarrow \infty$  (without the finite part the integral would be divergent because of the factor  $x_L = r^l n_L$  in the integrand and the fact that the pseudotensor  $\bar{\tau}^{\mu\nu}$  is not of compact support).

The result (5.10)–(5.11) permits us to define a very convenient notion of the *source* multipole moments (by opposition to the *radiative* moments defined below). Quite naturally, the source moments are constructed from the ten components of the tensorial function  $\mathcal{F}_L^{\mu\nu}(u)$ . Among these components four can be eliminated using the harmonic gauge condition (5.2), so in the end we find only six independent source multipole moments. Furthermore, it can be shown that by changing the harmonic gauge in the exterior zone one can further reduce the number of independent moments to only two. Here we shall report the result for the ‘main’ multipole moments of the source, which are the mass-type moment  $I_L$  and current-type  $J_L$  (the other moments play a small role starting only at highorder in the post-Newtonian expansion). We have [14]

$$I_L(u) = \text{finite part} \int d^3\mathbf{x} \int_{-1}^1 dz \left\{ \delta_l \hat{x}_L \Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)} \delta_{l+1} \hat{x}_{iL} \partial_t \Sigma_i \right. \\ \left. + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)} \delta_{l+2} \hat{x}_{ijL} \partial_t^2 \Sigma_{ij} \right\}, \quad (5.14)$$

$$J_L(u) = \text{finite part} \int d^3\mathbf{x} \int_{-1}^1 dz \varepsilon_{ab<i_l} \left\{ \delta_l \hat{x}_{L-1>a} \Sigma_b \right. \\ \left. - \frac{2l+1}{c^2(l+2)(2l+3)} \delta_{l+1} \hat{x}_{L-1>ac} \partial_t \Sigma_{bc} \right\}. \quad (5.15)$$

Here the integrand is evaluated at the instant  $u + z|\mathbf{x}|/c$ ,  $\varepsilon_{abc}$  is the Levi-Civita symbol,  $\langle L \rangle$  is the STF projection, and we employ the notation

$$\Sigma = \frac{\bar{\tau}^{00} + \bar{\tau}^{ii}}{c^2}; \quad \Sigma_i = \frac{\bar{\tau}^{0i}}{c}; \quad \Sigma_{ij} = \bar{\tau}^{ij} \quad (5.16)$$

(with  $\bar{\tau}^{ii} = \delta_{ij} \bar{\tau}^{ij}$ ). The multipole moments  $I_L, J_L$  are valid formally up to any post-Newtonian order, and constitute a generalization in the nonlinear theory of the usual mass and current Newtonian moments (see, [14] for details). It can be checked that, when considered at the 1PN order, these moments agree with the different expressions obtained in [9] (case of mass moments) and in [10] (current moments).

### C. Radiative moments

In linearized theory, where we can neglect the gravitational source term  $\Lambda^{\mu\nu}$  in (5.4), as well as the first term in (5.10), the source multipole moments coincide with the so-called radiative multipole moments, defined as the coefficients of the multipole expansion of the  $1/r$  term in the distance to the source at retarded times  $t - r/c = \text{constant}$ . However, in full nonlinear theory, the first term in (5.10) will bring another contribution to the  $1/r$  term at future null infinity. Therefore, the source multipole moments are not the ‘measured’ ones at infinity, and so they must be related to the real observables of the field at infinity which are constituted by the radiative moments. It has been known for a long time that the harmonic coordinates do not belong to the class of Bondi coordinate systems at infinity, because the expansion of the harmonic metric when  $r \rightarrow \infty$  with  $t - r/c = \text{constant}$  involves, in addition to the normal powers of  $1/r$ , some powers of the *logarithm* of  $r$ . Let us change the coordinates from harmonic to some Bondi-type or ‘radiative’ coordinates  $(\mathbf{X}, T)$  such that the metric admits a power-like expansion without logarithms when  $R \rightarrow \infty$  with  $T - R/c = \text{constant}$  and  $R = |\mathbf{X}|$  (it can be shown that the condition to be satisfied by the radiative coordinate system is that the retarded time  $T - R/c$  becomes asymptotically null at infinity). For the purpose of deriving the formula (5.20) below it is sufficient to transform the coordinates according to

$$T - \frac{R}{c} = t - \frac{r}{c} - \frac{2GM}{c^3} \ln \left( \frac{r}{r_0} \right), \quad (5.17)$$

where  $M$  denotes the ADM mass of the source and  $r_0$  is a gauge constant. In radiative coordinates it is easy to decompose the  $1/R$  term of the metric into multipoles and to define in that way the radiative multipole moments  $U_L$  (mass-type; where  $L = i_1 \dots i_l$  with  $l \geq 2$ ) and  $V_L$  (current-type; with  $l \geq 2$ ). (Actually, it is often simpler to bypass the need for transforming the coordinates from harmonic to radiative by considering directly the TT projection of the spatial components of the harmonic metric at infinity.) The formula for the definition of the radiative moments is

$$h_{ij}^{\text{TT}} = -\frac{4G}{c^2 R} \mathcal{P}_{ijab}(\mathbf{N}) \sum_{l=2}^{+\infty} \frac{1}{c^l l!} \left\{ N_{L-2} U_{abL-2}(T - R/c) - \frac{2l}{c(l+1)} N_{cL-2} \varepsilon_{cd(a} V_{b)dL-2}(T - R/c) \right\} + O\left(\frac{1}{R^2}\right) \quad (5.18)$$

where  $\mathbf{N}$  is the vector  $N_i = N^i = X^i/R$  (for instance  $N_{L-2} = N_{i_1} \dots N_{i_{l-2}}$ ), and  $\mathcal{P}_{ijab}$  denotes the TT projector

$$\mathcal{P}_{ijab} = (\delta_{ia} - N_i N_a)(\delta_{jb} - N_j N_b) - \frac{1}{2}(\delta_{ij} - N_i N_j)(\delta_{ab} - N_a N_b). \quad (5.19)$$

In the limit of linearized gravity the radiative multipole moments  $U_L$ ,  $V_L$  agree with the  $l$ th time derivatives of the source moments  $I_L$ ,  $J_L$ . Let us give, without proof, the result for the expression of the radiative mass-quadrupole moment  $U_{ij}$  including relativistic corrections up to the 3PN or  $1/c^6$  order inclusively [12,13]. The calculation involves implementing explicitly a post-Minkowskian algorithm defined in [8] for the computation of the nonlinearities due to the first term of (5.10). We find ( $U \equiv T - R/c$ )

$$\begin{aligned} U_{ij}(U) = & M_{ij}^{(2)}(U) + 2 \frac{GM}{c^3} \int_0^{+\infty} dv M_{ij}^{(4)}(U - v) \left[ \ln\left(\frac{cv}{2r_0}\right) + \frac{11}{12} \right] \\ & + \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^{+\infty} dv [M_{a<i}^{(3)} M_{j>a}^{(3)}](U - v) - \frac{2}{7} M_{a<i}^{(3)} M_{j>a}^{(2)}(U) \right. \\ & \left. - \frac{5}{7} M_{a<i}^{(4)} M_{j>a}^{(1)}(U) + \frac{1}{7} M_{a<i}^{(5)} M_{j>a}(U) + \frac{1}{3} \varepsilon_{ab<i} M_{j>a}^{(4)} J_b(U) \right\} \\ & + 2 \left( \frac{GM}{c^3} \right)^2 \int_0^{+\infty} dv M_{ij}^{(5)}(U - v) \\ & \times \left[ \ln^2\left(\frac{cv}{2r_0}\right) + \frac{57}{70} \ln\left(\frac{cv}{2r_0}\right) + \frac{124627}{44100} \right] \\ & + O\left(\frac{1}{c^7}\right). \end{aligned} \quad (5.20)$$

The superscript  $(n)$  denotes  $n$  time derivations. The quadrupole moment  $M_{ij}$  entering this formula is closely related to the source quadrupole  $I_{ij}$ ,

$$M_{ij} = I_{ij} + \frac{2G}{3c^5} \{ K^{(3)} I_{ij} - K^{(2)} I_{ij}^{(1)} \} + O\left(\frac{1}{c^7}\right), \quad (5.21)$$

where  $K$  is the Newtonian moment of inertia (see equation (4.24) in [27]; we are using here a mass-centred frame so that the mass-dipole moment  $I_i$  is zero). The Newtonian term in (5.20) corresponds to the quadrupole formalism. Next, there is a quadratic nonlinear correction term with multipole interaction  $M \times M_{ij}$  which represents the effect of tails of gravitational waves (scattering of linear waves off the spacetime curvature generated by the mass  $M$ ). This correction is of order  $1/c^3$  or 1.5PN and takes the form of a non-local integral with logarithmic kernel [30]. It is responsible notably for the term proportional to  $\pi\tau^{1/4}$  in the formula for the phase (6.13) below. The next correction, of order  $1/c^5$  or 2.5PN, is constituted by quadratic interactions between two mass-quadrupoles, and between a mass-quadrupole and the constant current dipole [12]. This term contains also a non-local integral, which is due to the radiation of gravitational waves by the distribution of the stress-energy of linear waves [31,32,30,12]. Finally, at the 3PN order in (5.20) the first cubic nonlinear interaction appears, which is of the type  $(M \times M \times M_{ij})$  and corresponds to the tails generated by the tails themselves [13].

## VI. INSPIRAL BINARY WAVEFORM

To conclude, let us give (without proof) the result for the two polarization waveforms  $h_+(t)$  and  $h_\times(t)$  of the inspiralling compact binary developed to 2PN order in the amplitude and to 2.5PN order in the phase. The calculation

was done by Blanchet, Damour, Iyer, Will and Wiseman [25,26,15,27,33], based on the formalism reviewed in section V and, independently, on that defined in [15]. Following [33] we present the polarization waveforms in a form which is ready for use in the data analysis of binary inspirals in the detectors VIRGO and LIGO (the analysis will be based on the optimal filtering technique reviewed in section II). We find, extending the Newtonian formulae in section III,

$$h_{+, \times} = \frac{2G\mu}{c^2 R} \left( \frac{Gm\omega}{c^3} \right)^{2/3} \times \{ H_{+, \times}^{(0)} + x^{1/2} H_{+, \times}^{(1/2)} + x H_{+, \times}^{(1)} + x^{3/2} H_{+, \times}^{(3/2)} + x^2 H_{+, \times}^{(2)} \}, \quad (6.1)$$

where the various post-Newtonian terms, ordered by  $x$ , are given for the plus polarization by

$$H_+^{(0)} = -(1 + c_i^2) \cos 2\psi, \quad (6.2)$$

$$H_+^{(1/2)} = -\frac{s_i}{8} \frac{\delta m}{m} [(5 + c_i^2) \cos \psi - 9(1 + c_i^2) \cos 3\psi], \quad (6.3)$$

$$H_+^{(1)} = \frac{1}{6} [(19 + 9c_i^2 - 2c_i^4) - \nu(19 - 11c_i^2 - 6c_i^4)] \cos 2\psi - \frac{4}{3} s_i^2 (1 + c_i^2) (1 - 3\nu) \cos 4\psi, \quad (6.4)$$

$$H_+^{(3/2)} = \frac{s_i}{192} \frac{\delta m}{m} \{ [(57 + 60c_i^2 - c_i^4) - 2\nu(49 - 12c_i^2 - c_i^4)] \cos \psi - \frac{27}{2} [(73 + 40c_i^2 - 9c_i^4) - 2\nu(25 - 8c_i^2 - 9c_i^4)] \cos 3\psi + \frac{625}{2} (1 - 2\nu) s_i^2 (1 + c_i^2) \cos 5\psi \} - 2\pi(1 + c_i^2) \cos 2\psi, \quad (6.5)$$

$$H_+^{(2)} = \frac{1}{120} [(22 + 396c_i^2 + 145c_i^4 - 5c_i^6) + \frac{5}{3}\nu(706 - 216c_i^2 - 251c_i^4 + 15c_i^6) - 5\nu^2(98 - 108c_i^2 + 7c_i^4 + 5c_i^6)] \cos 2\psi + \frac{2}{15} s_i^2 [(59 + 35c_i^2 - 8c_i^4) - \frac{5}{3}\nu(131 + 59c_i^2 - 24c_i^4) + 5\nu^2(21 - 3c_i^2 - 8c_i^4)] \cos 4\psi - \frac{81}{40} (1 - 5\nu + 5\nu^2) s_i^4 (1 + c_i^2) \cos 6\psi + \frac{s_i}{40} \frac{\delta m}{m} \{ [11 + 7c_i^2 + 10(5 + c_i^2) \ln 2] \sin \psi - 5\pi(5 + c_i^2) \cos \psi - 27[7 - 10 \ln(3/2)](1 + c_i^2) \sin 3\psi + 135\pi(1 + c_i^2) \cos 3\psi \}, \quad (6.6)$$

and for the cross-polarization by

$$H_\times^{(0)} = -2c_i \sin 2\psi, \quad (6.7)$$

$$H_\times^{(1/2)} = -\frac{3}{4} s_i c_i \frac{\delta m}{m} [\sin \psi - 3 \sin 3\psi], \quad (6.8)$$

$$H_\times^{(1)} = \frac{c_i}{3} [(17 - 4c_i^2) - \nu(13 - 12c_i^2)] \sin 2\psi - \frac{8}{3} (1 - 3\nu) c_i s_i^2 \sin 4\psi, \quad (6.9)$$

$$H_\times^{(3/2)} = \frac{s_i c_i}{96} \frac{\delta m}{m} \{ [(63 - 5c_i^2) - 2\nu(23 - 5c_i^2)] \sin \psi - \frac{27}{2} [(67 - 15c_i^2) - 2\nu(19 - 15c_i^2)] \sin 3\psi + \frac{625}{2} (1 - 2\nu) s_i^2 \sin 5\psi \} - 4\pi c_i \sin 2\psi, \quad (6.10)$$

$$H_\times^{(2)} = \frac{c_i}{60} [(68 + 226c_i^2 - 15c_i^4) + \frac{5}{3}\nu(572 - 490c_i^2 + 45c_i^4) - 5\nu^2(56 - 70c_i^2 + 15c_i^4)] \sin 2\psi + \frac{4}{15} c_i s_i^2 [(55 - 12c_i^2) - \frac{5}{3}\nu(119 - 36c_i^2) + 5\nu^2(17 - 12c_i^2)] \sin 4\psi - \frac{81}{20} (1 - 5\nu + 5\nu^2) c_i s_i^4 \sin 6\psi - \frac{3}{20} s_i c_i \frac{\delta m}{m} \{ [3 + 10 \ln 2] \cos \psi + 5\pi \sin \psi - 9[7 - 10 \ln(3/2)] \cos 3\psi - 45\pi \sin 3\psi \}. \quad (6.11)$$

The notation is consistent with sections III and IV. In particular, the post-Newtonian parameter  $x$  is defined by (4.13). We use the shorthands  $c_i = \cos i$  and  $s_i = \sin i$  where  $i$  is the inclination angle. The basic phase variable  $\psi$  entering the waveforms is defined by

$$\psi = \phi - \frac{2Gm\omega}{c^3} \ln \left( \frac{\omega}{\omega_0} \right), \quad (6.12)$$

where  $\phi$  is the actual orbital phase of the binary, and where  $\omega_0$  can be chosen as the seismic cut-off of the detector (see [33] for details). As for the phase evolution  $\phi(t)$ , it is given up to 2.5PN order, generalizing the Newtonian formula (4.19), by

$$\begin{aligned} \phi(t) = \phi_0 - \frac{1}{\nu} & \left\{ \tau^{5/8} + \left( \frac{3715}{8064} + \frac{55}{96} \nu \right) \tau^{3/8} - \frac{3}{4} \pi \tau^{1/4} \right. \\ & + \left( \frac{9275495}{14450688} + \frac{284875}{258048} \nu + \frac{1855}{2048} \nu^2 \right) \tau^{1/8} \\ & \left. + \left( -\frac{38645}{172032} - \frac{15}{2048} \nu \right) \pi \ln \tau \right\}, \end{aligned} \quad (6.13)$$

where  $\phi_0$  is a constant and where we recall that the dimensionless time variable  $\tau$  was given by (4.17). The frequency is equal to the time derivative of (6.13), hence

$$\begin{aligned} \omega(t) = \frac{c^3}{8Gm} & \left\{ \tau^{-3/8} + \left( \frac{743}{2688} + \frac{11}{32} \nu \right) \tau^{-5/8} - \frac{3}{10} \pi \tau^{-3/4} \right. \\ & + \left( \frac{1855099}{14450688} + \frac{56975}{258048} \nu + \frac{371}{2048} \nu^2 \right) \tau^{-7/8} \\ & \left. + \left( -\frac{7729}{21504} - \frac{3}{256} \nu \right) \pi \tau^{-1} \right\}. \end{aligned} \quad (6.14)$$

We have checked that both waveforms (6.2)–(6.7) and phase/frequency (6.13)–(6.14) agree in the test mass limit  $\nu \rightarrow 0$  with the results of linear black hole perturbations as given by Tagoshi and Sasaki [34].

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